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# MTHE 280 - Lecture Notes
## Advanced Calculus

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# Contents

# 1 Introduction to Multivariable Functions

A function $f(x, y)$ is a rule that assigns to every element $x$ a unique element $y$, and is denoted by $f : x \to y$, where $x$ is the domain of $f$ and $y$ is the codomain of $f$

**Example**

$f : \mathbf{N} \to \mathbf{R}, f(x) = 2x$

In this case, every value of $f$ is even and does not take the whole codomain

We introduce the range, a subset of the codomain, $range(f) \subseteq codomain(f)$

## 1.1 Properties of functions

**One-one/Injective**

$f : X \to Y$ if $x_1, x_2 \in X, f(x_1) = f(x_2)$

**Onto/Surjective**

$f : X \to Y$ is onto if for every $y \in Y$, there exists some $x \in X$ such that $f(x) = y$

In this case, codomain = range

**Bijective**

if $f : x \to y$ is both one-one and onto, it is bijective

**Scalar-valued**

Consider $f : x \to y$ where $x \subseteq \mathbf{R}$ and $y \subseteq \mathbf{R}$, $n, m \in \mathbf{N}$

When the codomain is just $\mathbf{R}$, the function is called a Scalar-valued function

**Example**

$f : \mathbf{R^2} \to \mathbf{R}$ where $f(x, y) = \sqrt{x^2 + y^2}$

This returns the length of a 2D vector, which is a scalar

**Vector-valued**

A vector-valued function has codomain $\mathbf{R^n}$ where $n > 1, n \in N$

**Example**

$f : \mathbf{R} \to \mathbf{R^2}$, $f(x) = (\cos x, \sin x)$

## 1.2 Identify domain and codomain

**Examples**

$f(x) = \ln x$, domain = $(0, \infty)$, codomain = $\mathbf{R}$

$f(x) = \sqrt{2 - x}$, domain = $(-\infty, 2]$, codomain = $(0, \infty)$

$f(x, y) = (\sqrt{1 - x^2 - y^2}, \ln(y + 1), x^2 + y^2)$

1: $x^2 + y^2 = 1$ 2: $y > -1$

domain: $\{(x, y) \in \mathbf{R^2} : x^2 + y^2 \leq 1, y > -1\}$

# 2 Level Curves and Contours

### Level Curve

Given a scalar-valued function, the level curve at height $c$ is the curve in $\mathbf{R^2}$ s.t. $f(x, y) = c$

Or, the level curve at height $c = \{(x, y) \in \mathbf{R^2} | f(x, y) = c\}$

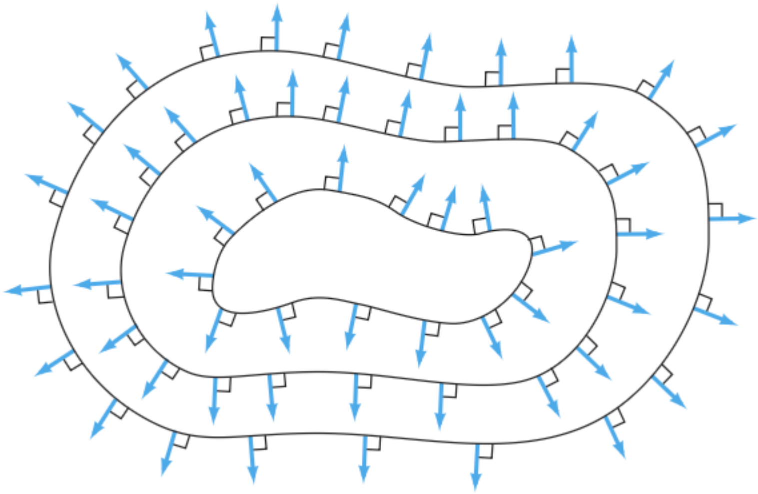


Figure 3.31 A gradient vector field $\mathbf{F} = \nabla f$. Equipotential lines are shown where $f$ is constant.

### Contour

The contour curve at height $c$ is the collection of points $(x, y, z)$ s.t. $z = f(x, y) = c$

Or, $\{(x, y, z) \in \mathbf{R^3} | z = f(x, y) = c\}$

The projection of the contour is the level curve

### Section

A section of a surface by a place is just the intersection of the surface with that plane

# 3 Limits of a function

General form: $f : \mathbf{R} \to \mathbf{R}$

$\lim_{x \to a} f(x) = L, \therefore f(x)$ tends to $L$ as $x$ tends to $a$

## 3.1 L'Hospital's Rule

If we have a case where we are evaluating a limit and we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can use $\lim_{x \to a} \frac{f'(x)}{g'(x)}$

Why?: The ratio $\frac{f(x)}{g(x)}$ near $a$ depends not only on the values of $f$ and $g$, but on how fast they approach 0 or $\infty$

## 3.2 Limits in two variables

Let $f : \mathbf{R^2} \to \mathbf{R}$, $\lim_{(x,y) \to (a,b)} f(x, y)$

**The Line $y = mx$ trick**

All paths approaching point (e.g. $(0, 0)$) must give the same value

A simple test path is a straight line $mx$ through the origin, and plug $f(x, y) \to f(x, mx)$

**If the result depends on** $m$, the limit does not exist

**Does Exist Example**

$$\lim_{(x,mx) \to (0,0)} \frac{x^2}{x^2 + y^4}$$

$$\lim_{x \to 0} \frac{x^2}{x^2 + m^4 x^4}$$

$$\lim_{x \to 0} \frac{1}{1 + m^4 x^2} = 1 \therefore \text{limit exists}$$

**Does Not Exist Example**

$$\lim_{(x,y) \to (0,0)} \frac{1}{1 + m^2} = \frac{x^2}{x^2 + m^2 x^2} = \frac{1}{1 + m^2} \therefore \text{limit does not exist}$$

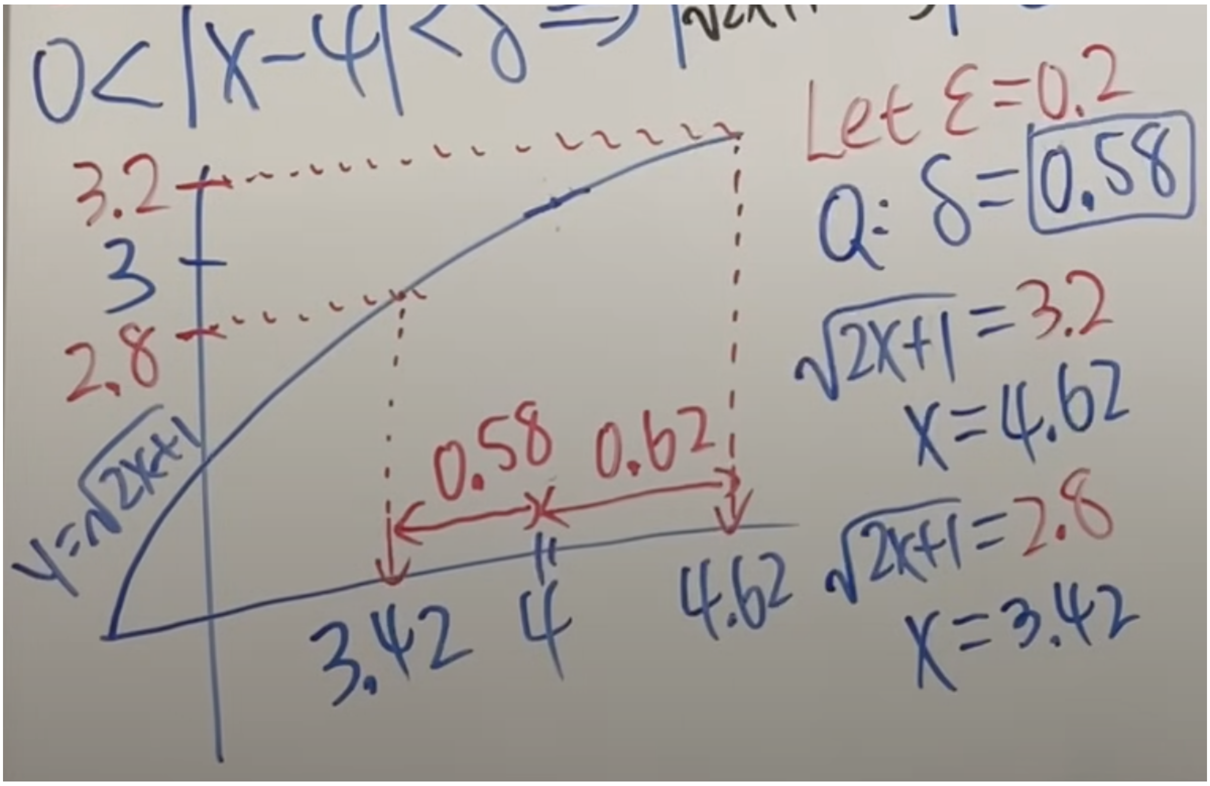
## 3.3 Epsilon-delta definition of a limit

$\lim_{x \to a} f(x) = L$ means $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

**Example:** we know that $\lim_{x \to 4} \sqrt{2x + 1} = 3$ by plugging in 4 into the continuous function

To prove this, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - 4| < \delta \Rightarrow |\sqrt{2x + 1} - 3| < \delta$

If $x$ is near 4, of a distance less than $\delta$, then the corresponding value of the function is near the limit $L = 3$, of a distance $\varepsilon$




### 3.3.1 General solution process

**Proof:** Given $\varepsilon > 0$ We want to find $\delta > 0$ such that if $0 < ||x-a|| < \delta$, then $|f(x)-L| < \varepsilon$

Start with $|f(x) - L|$ and manipulate it to relate it to $||x - a||$ For instance, show: $|f(x) - L| \leq c||x - a||$ for some $c > 0$

Choose $\delta = \frac{\varepsilon}{c}$ and show that $|f(x) - L| < c||x - a|| < c\delta = \varepsilon$

Therefore, $\lim_{x \to a} f(x) = L$

**Triangle Inequality**

It says: $|a + b| \leq |a| + |b|$

**Order Trick**

Ex: $\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = 0$, lim is likely to exist when order is $\geq 1$, here it is 1

**Simplify Trick**

We can: $\frac{3|x|y^2}{x^2+y^2} \leq \frac{3|x|y^2}{y^2} = 3|x|$

We can also: $|x| \leq \sqrt{x^2 + y^2}$

**Linear combination of coordinate differences**

$|a(x - a) + b(y - b)| \leq |a||x - a| + |b||y - b| \leq (|a| + |b|)||\mathbf{x} - \mathbf{a}||.$

## 3.4 When to use either strategy

We use the epsilon-delta proof to rigorously prove that a limit exists (or equals some value)

We take the limit along lines, parabolas, or curves to test whether a limit exists, or to guess its value. It is useful when you are not sure if the limit exists.

## 3.5 $\varepsilon - \delta$ for vector-valued functions

Let $F : U(\subseteq \mathbf{R^n}) \to \mathbf{R^m}, \vec{a} \in U$

We write $\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = \vec{L}, \forall\varepsilon > 0, \exists\delta > 0$ s.t. $||F(\vec{x}) - \vec{L}|| < \varepsilon$ if $||\vec{x} - \vec{a}|| < \delta$

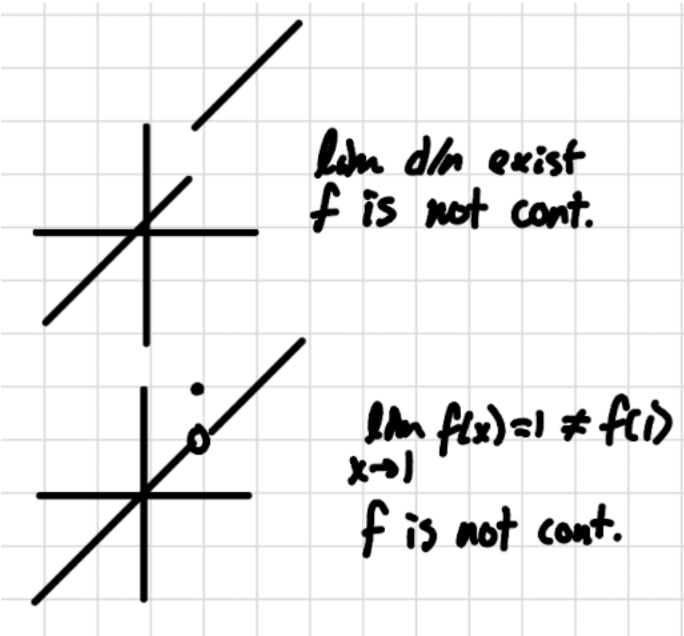
Ex: does $\lim_{(x,y)\to(0,0)} (\frac{3xy^2}{x^2+y^2}, \frac{e^x+\cos y}{x^2+y^2+1})$ exist?

We know that the first component does. For the second component, both the numerator and the denominator are continuous at $(0, 0)$, thus we can plug in that point and get that the limit approaches 2

# 4 Continuity and its properties

## 4.1 Continuity of single variable functions

Let $f : A \to R, a \in A$. $f$ is continuous if (1) $\lim_{x\to a} f(x)$ exists and (2) $\lim_{x\to a} f(x) = f(a)$




## 4.2 Continuity of multivariable functions

Let $f : U(\subseteq \mathbf{R^n} \to \mathbf{R}$ and $\vec{a} \in U$. $f$ is continuous at $\vec{a}$ if (1) $\lim_{\vec{x}\to\vec{a}} F(\vec{x})$ exists and (2) $\lim_{\vec{x}\to\vec{z}} F(\vec{x}) = F(\vec{a})$

## 4.3 Properties of continuity (scalar- and vector-valued functions)

Suppose that $f$ and $g$ are continuous at $\vec{a} \in U$

1. $f + g$ is continuous at $\vec{a}$
2. $f * g$ is continuous at $\vec{a}$
3. $\frac{f}{g}$ is continuous at $\vec{a}$ if $g(\vec{a}) \neq 0$

Further:

1. $\lim_{\vec{x}\to\vec{a}}(f + g)(\vec{x}) = f(\vec{a}) + g(\vec{a})$
2. $\lim_{\vec{x}\to\vec{a}}(f * g)(x) = f(\vec{a})g(\vec{a})$
3. $\lim_{\vec{x}\to\vec{a}} \left(\frac{f}{g}\right)(\vec{x}) = \frac{f(\vec{a})}{g(\vec{a})}$ if $g(\vec{a}) \neq 0$

**Example:**

$$f(x) = \begin{cases} \frac{3xy^2}{x^2+y^2}, (x, y) \neq (0, 0), \\ \\ a, (x, y) = (0, 0). \end{cases}$$

**For which values of $a$ is $F$ continuous?**

We know that the first component is continuous everywhere, except possible at $(0, 0)$

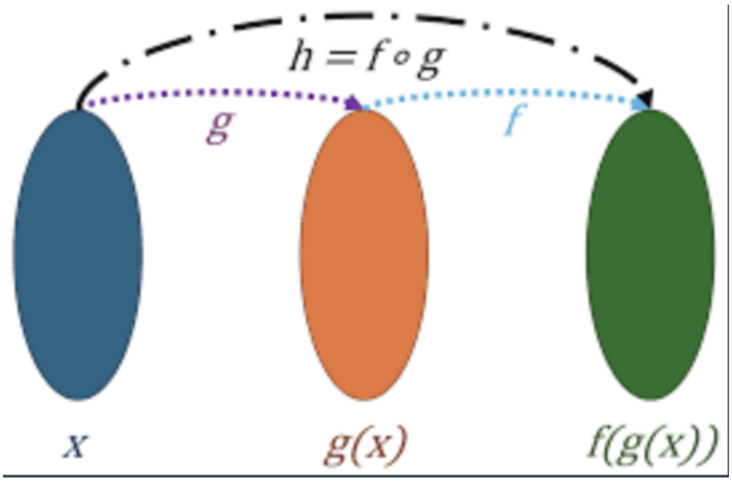
For continuity at $(0, 0)$, we need the limit of $F$ at $(0, 0) =$ a, which is equivalent to saying that the continuous function $F(0, 0) = a$

That means we need to compute the first term's limit while approaching $(0, 0)$, which is $= 0$

$\therefore a = 0$

## 4.4 Composition of two continuous functions

If: 1. $g$ is continuous at $x = a$, and 2. $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$, where $f(g(x)) \to f(g(a))$



# 5 Differentiation of multivariable functions

## 5.1 The derivative

$f$ is differentiable at $c$ if $\lim_{h \to c} \frac{f(x+h)-f(c)}{h}$ exists. If the limit exists, then it is denoted by $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(h)}{h}$, where $f'(x)$ captures the rate of change of $f$ near $c$

If $f(c)$ exists, we can draw a tangent line at $c$, and its slope is $f'(c)$

## 5.2 Notation

An **open ball** in $\mathbf{R^n}$ with centre at $\vec{a} \in \mathbf{R^n}$ and radius $r : B(\vec{a}, r)$. The ball is open, meaning that the boundary points are not included

**Definition:** A point $\vec{a}$ is an **interior point** of a set $A$ if there exists an open ball $B_\varepsilon(\vec{a})$, for some $\varepsilon > 0$, such that $B_\varepsilon(\vec{a}) \subseteq A$. So, the open ball lies entirely inside the set, without touching its complement

**Definition**: A **boundary point** is a point $\vec{a}$ such that every open ball $B_\varepsilon(\vec{a})$, no matter how small $\varepsilon > 0$ is, intersects the function and its complement (not the function)

Essentially, an open ball is all points strictly inside a certain radius form the centre, not including the edge. The interior points are inside the open ball, and boundary points are on the edge.

A set $U \subseteq \mathbf{R^n}$ is called open if every point of $U$ is an interior point

## 5.3 Partial Differentiation

$f$ is partially differentiable wrt $x$ at $(a, b)$ if $\lim_{x \to a} \frac{f(a+h,h)-f(a,h)}{h}$ exists. If exists: $\frac{\partial f}{\partial x}(a, b)$ or $f_x(a, b)$

# 6 Partial Differentiation (cont.)

## 6.1 Tangent plane visualized

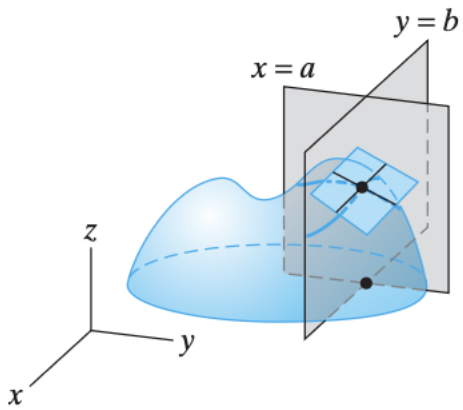



Figure 2.51 The tangent plane at $(a, b, f(a, b))$ contains the lines tangent to the curves formed by intersecting the surface $z = f(x, y)$ by the planes $x = a$ and $y = b$.

## 6.2 Directional derivative

The directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $p$ in the direction of a vector $\vec{v}$ is the rate at which $f$ changes at $p$ as you move in the direction of $\vec{v}$

$D_{\vec{v}}f(p) = \nabla f(p) \cdot \vec{v}$

For vector valued functions, we can compute using the Jacobian $D_{\vec{v}}f(p) = DH(p) \cdot \vec{v}$

**Definition:** The directional derivative of $f$ at $\vec{a} = (a, b)$ in the direction of $\vec{v}$ is given by $D_{\vec{v}}f(\vec{a}) = \lim\limits_{h\to 0} \frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h}$, if it exists

**Example:** let $f(x, y) = x^2y - 3x$, $D_{\vec{v}}f(0, 0) =?$ where $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$D_{\vec{v}}f(0, 0) = \lim\limits_{h\to 0} \frac{f(0,0)+h\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)-f(0,0)}{h}$

Simplify, then plug in h

$= -\frac{3}{\sqrt{2}}$

## 6.3 Multivariable differentiability at $(a, b)$

**Definition:** $f : \mathbf{R}^2 \to \mathbf{R}$ is differentiable at $(a, b)$ if $\exists h(x, y) = f(a, b) + f_x(a, b) + f_y(a, b)$

1. $f_x(a, b)$ and $f_y(a, b)$ exists

2. $\exists \mathbf{R} f'(a)$ s.t. $\lim_{h \to 0} \frac{f(x)-h(x,y)}{|x-a|} = 0$, where $h(x,y)$ is the equation of the tangent plane (or line) $f(a,b) + f_x(a,b)(x-a) + f_y(a,b))y - b$

**How?**

**Single variable differentiability** is defined by $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

We can rearrange to emphasize linear approximation: $\lim_{x \to a} \frac{f(x)-[f(a)+f'(a)(x-a)]}{x-a} = 0$

This is saying that the function is differentiable at $a$ if it can be approximated by th linear function $h(x,y)$ with error smaller than order $|x-a|$

**Multivariable differentiability** is now as follows $\lim_{(x,y) \to (a,b)} \frac{f(x,y)-h(x,y)}{||((x,y)-(a,b)||} = 0$

# 7 Gradients, More Derivatives, and the Jacobian

## 7.1 Gradient

The gradient of a scalar function is a vector that collects all the partial derivatives of $f$ with respect to each variable:

$\nabla f = (f_{x_1}, f_{x_2}, \dots, f_{x_n})$

At a specific point, the gradient becomes:

$\nabla f(\vec{a}) = (f_{x_1}(\vec{a}), \dots, f_{x_n}(\vec{a}))$

This vector points in the direction of the steepest increase of $f$ and its magnitude gives the rate of increase

The difference vector:

$\vec{x} - \vec{a} = (x_1 - a_1, \dots, x_n - a_n)$

The linear approximation of $f$ near $\vec{a}$ can be written as:

$\nabla f(\vec{a})(\vec{x} - \vec{a}) = f_{x_1}(\vec{a})(x_1 - a_1) + \cdots + f_{x_n}(\vec{a})(x_n - a_n)$

**Example:**

Let $f(x, y) = xy^2 + e^{xy}$, find the gradient at $(0, 0)$

$f_x = y^2 + ye^{xy}, f_y = 2yx + xe^{xy}$

$\nabla f = (f_x, f_y) = (y^2 + ye^{xy}, 2xy + xe^{xy})\ \nabla f(0, 0) = (0, 0)$

**Dot product of two vectors**

If $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n$

## 7.2 Derivative Matrix

Let $U \subseteq \mathbf{R^n}$ and $f : U(\subseteq \mathbf{R^n}) \to \mathbf{R^m}$

$f = (f_1, f_2, \dots, f_m)$

Let $f(x, y) = (x^2, x + y)$

$f_1(x) = x^2, f_2(x) = x + y$

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \cdots \\ \nabla f_m \end{matrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \dots, \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \dots, \frac{\partial f_2}{\partial x_n} \\ \cdots \\ \frac{\partial f_m}{\partial x_1}, \frac{\partial f_m}{\partial x_2}, \dots, \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

This matrix is called the matrix of partial derivatives of $f$, otherwise called the Derivative Matrix or the **Jacobian Matrix**. Essentially, the derivative is a linear map, and in coordinates it is built from the partial derivatives

**Example:**

Let $f(x, y) = (xy, y^2 \sin x, x^3 e^y)$, find the derivative matrix

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{matrix} = \begin{matrix} y, x \\ y^2 \cos x, 2y \sin x \\ 3x^2 e^y, x^3 e^y \end{matrix}$$

## 7.3 Differentiability in higher dimensions $f : U \to \mathbf{R^m}$

$f$ is differentiable if: - $Df(\vec{a})$ exists - Tangent plane $h : \mathbf{R^n} \to \mathbf{R^m}$, $h(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$, where $Df(\vec{a})(\vec{x} - \vec{a})$ is a matrix multiplication, satisfies $\lim\limits_{\vec{x}\to\vec{a}} \frac{||f(\vec{x})-h(\vec{x})||}{||\vec{x}-\vec{a}||} = 0$, which is hard to use

This is why we introduce the following theorems:

### 7.3.1 Theorems for higher-dimension differentiability

**Theorem 1:**

If $f = (f_1, f_2, \ldots, f_m)$, then $f$ is differentiable at $\vec{a} \Leftrightarrow f_1, f_2, \ldots, f_m$ is differentiable at $\vec{a}$

**Theorem 2:**

If $f = (f_1, f_2, \ldots, f_m)$ and all partials $\frac{\partial f_i}{\partial x_j}$, as $i, j, \ldots, i_m, j_m$, are continuous then $f$ is differentiable

**Example:**

$f(x, y) = (x^2 y, e^y \sin x)$ is differentiable because all of its partial derivatives are continuous

**Theorem 3:**

If $f$ is differentiable at $\vec{a}$, then directional derivatives can be computed using: $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$

If $f$ is differentiable at $\vec{a}$, then $D_{\vec{v}} f(\vec{a}) = Df(\vec{a})\vec{v}$ where $Df(\vec{a})\vec{v}$ is a matrix multiplication

**Example:**

$f(x, y) = (e^x y, x^2 y)$, find rate of change of $f$ at $(1, 2)$ in direction $\vec{v} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$Df = \begin{matrix} e^x y, & e^x \\ 2xy, & x^2 \end{matrix}, \; Df(1,2) = \begin{matrix} 2e, & e \\ 4, & 1 \end{matrix}$$

$$Df(1,2)\vec{v} = \begin{matrix} 2e, & e \\ 4, & 1 \end{matrix} \cdot \begin{matrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{matrix} = \begin{matrix} e + \frac{\sqrt{3}}{2} e \\ 2 + \frac{\sqrt{3}}{2} \end{matrix}$$

## 7.4 Properties of Differentiability

Let $F : \mathbf{R^n} \to \mathbf{R}, G : \mathbf{R^n} \to \mathbf{R}$ be differentiable at $\vec{a}$

- $F + G$ is differentiable at $\vec{a}$
- $F \cdot G$ is differentiable at $\vec{a}$
- If $G(\vec{a}) \neq 0$, $\frac{F}{G}$ is differentiable at $\vec{a}$
- If $f$ is differentiable at $a$ and $g$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at $a$ and $\frac{d}{dx}(g \circ f) = g'(f(a)) * f'(a)$

- The graph of a function is the set $\{(x, y, f(x, y)) \in \mathbf{R^3} : (x, y) \in \text{ domain}\}$
- If $f_x, f_y, f_{xy}, f_{yx}$ are continuous, then $f_{xy} = f_{yx}$

# 8 Differentiability in higher dimension

## 8.1 Chain Rule in Composition

$D(G \circ F)(\vec{a}) = DG(F(\vec{a}))DF(\vec{a})$, where the RHS is a matrix multiplication

**Example:** $F(x, y) = (x^2y, e^{3x})$ and $G(x, y) = (x + y, xy, \sin(2x - y))$

Find: $D(G \circ F)(1, 1)$, where $(1, 1) = (\vec{a})$

Apply the chain rule equation and get $= DG(1, e^3)DF(1, 1)$

$$DF = \begin{matrix} 2xy & x^2 \\ 3e^{3x} & 0 \end{matrix} \text{ and } DG = \begin{matrix} 1 & 1 \\ y & x \\ 2\cos(2x - y) & -\cos(2x - y) \end{matrix}$$

$$DF(1, 1) = \begin{matrix} 2 & 1 \\ 3e^3 & 0 \end{matrix} \text{ and } DG(1, e^3) = \begin{matrix} 1 & 1 \\ e^3 & 1 \\ 2\cos(2 - e^3) & -\cos(2 - e^3) \end{matrix}$$

$$\text{Now, } D(G \circ F)(1, 1) = \begin{matrix} 2 + 3e^3 & 1 \\ 5e^3 & e^3 \\ 4\cos(2 - e^3) - 3e^3\cos(2 - e^3) & 2\cos(2 - e^3) \end{matrix}$$

## 8.2 Polar Coordinate Examples

$x = r\cos\theta, y = r\sin\theta$

$DH(r, \theta) = DG(r\cos\theta, r\sin\theta)DF(r, \theta)$

$DH(r, \theta) = \frac{\partial G}{\partial x}\cos\theta + \frac{\partial G}{\partial y}\sin\theta - \frac{\partial G}{\partial x}r\sin\theta + \frac{\partial G}{\partial y}\cos\theta$

**Example: Find DH**

With a given $r, \theta, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$, we can find $DH(r, \theta)$ through the chain rule

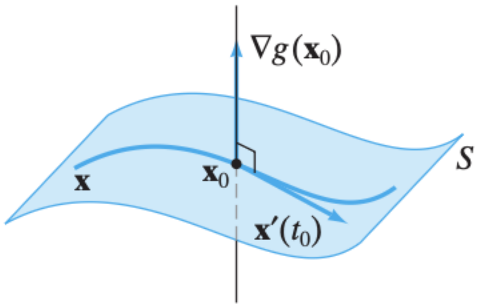
**Example: Find DG**

With a given $r, \theta, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial \theta}$, we can find $DG$ with: $\left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right] = \left[\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right] \cdot DF^{-1}$

# 9 Applications of the Gradient

## 9.1 Gradients and level curves

If we have a level curve for the function $x^2+y^2$, so $f(x,y)=c=x^2+y^2$, then the gradient $\nabla F$ is always perpendicular to the tangent plane to the level curve




Thus, the equation of the tangent plane is given by $\nabla F \cdot (\vec{x}-\vec{a}) = 0, \forall \vec{x}$ on tangent plane, where $\vec{a}$ is the fixed reference vector

**Example:** Find equation of tangent plane given the function and the reference vector

$f(x,y) = x^2y + ye^x$ at $(0,1,-1)$

Isolate and get the gradient: $f(x,y,z) = z - x^2y + ye^x$ $\nabla F = (-2xy + ye^x, -x^2 + e^x, 1)$ $\nabla F(0,1,-1) = (1,1,1)$

$(1,1,1)\cdot(x-0, y-1, z+1) = 0 \therefore x+y+z=0$

## 9.2 Magnitude of $\nabla F$

Consider the directional derivative $D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$

In what direction does the function increase the most?

If $\theta$ is the angle between $\vec{v}$ and teh gradient vector $\nabla f(\vec{a})$, then we have:

$D_{\vec{v}}f(\vec{a}) = ||\nabla f(a)||||\vec{v}|| \cos\theta = ||\nabla f(\vec{a})|| \cos\theta$ because the magnitude of the unit vector $\vec{v} = 1$

Thus, the max ROC is at $\theta = 0, = ||\nabla f(\vec{a})||$

The min ROC is at $\theta = \pi, = -||\nabla f(\vec{a})||$ and is opposite to $\nabla f(\vec{a})$

### 9.2.1 Example

Given $f(x,y) = 3\sin xy, \vec{a} = (1,\pi)$ find: 1. direction of max ROC, value of ROC at $f(\vec{a})$, and direction of tangent to the level curve at $\vec{a}$

1. Get gradient, plug in point, $\therefore$ max ROC is in the direction of gradient
2. Get magnitude of gradient at point, $\therefore$ this is the max ROC
3. $\nabla f$ is perpendicular to tangent line to the level curve at $(1,\pi)$. Find $\vec{v} \perp (-3\pi, -3)$

Method: change values in vector, change sign of 1

$\vec{v}_1 = (3, -3\pi)$ SOLVE USING CHAT

# 10 Conservative Vector Fields

A vector field is conservative if $\exists f : U \to \mathbf{R}$ such that $F = \nabla f$

The function $f$ is called a potential function of $F$

**Example:** $F(x, y) = (2x, 2y)$

Thus, if $F = \nabla f$ and the potential function $f(x, y) = x^2 + y^2$, then $F(x, y)$ is conservative and $f$ is the potential function

## 10.1 Test for conservative

Function $G(x, y, z)$ is conservative if

$$(G_1)_y = (G_2)_x \qquad (G_2)_z = (G_3)_y \qquad (G_1)_z = (G_3)_x$$

$$\parallel \qquad\qquad \parallel \quad \parallel \qquad\qquad \parallel \quad \parallel$$

$$F_{xy} = F_{yx} \qquad F_{yz} = F_{zy} \qquad F_{xz} = F_{zy}$$

## 10.2 Reconstruct a potential function given its gradient

Find $\nabla f = (f_x, f_y, f_z) = g = (g_1, g_2, g_3)$

1. Integrate $g_1$ wrt $x$

$f(x, y, z) = \int g_1 dx + h(y, z)$

2. Differentiate wrt $y$, set equal to $g_2$, solve for $h(y, z)$ by integrating wrt $y$ and get a $k(z)$ term
3. Differentiate wrt $z$, set equal to $g_3$, solve for $k(z)$ up to constant $C$
4. Assemble final $f(x, y, z) + C$

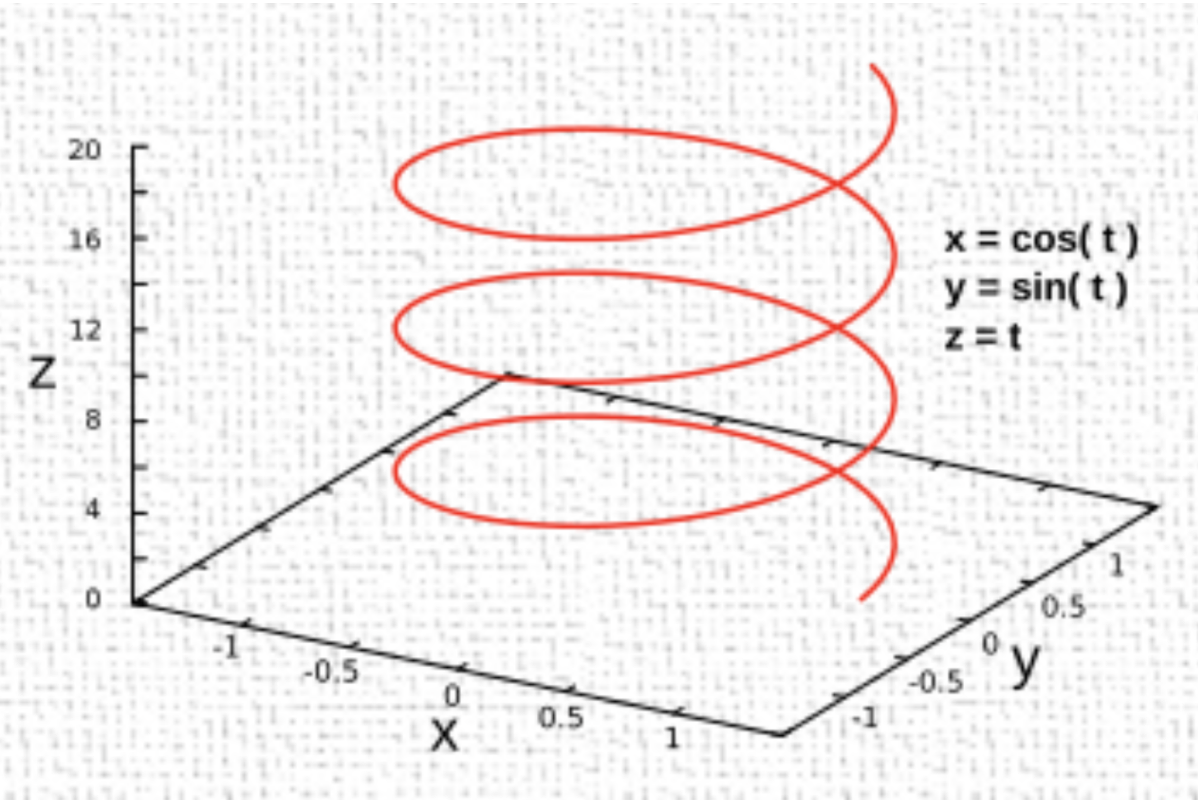
# 11 Parametric Equations and Class

**Definition of Path:** a continuous function $f : I \to \mathbf{R^n}$ where $I \in \mathbf{R}$ is on the interval $[a, b]$

## 11.1 Parametrization

$f(a)$ = starting point of $f$, $f(b)$ = end point of $f$

The Im of the path, denoted by $f(I)$ is called the curve in $\mathbf{R^2}$ and $f$ is a parametrization of $C$




**Important result:** Parametrization is not unique

$f(t) = (\cos t, \sin t)$ and $g(t) = (t, \sqrt{1 - t^2})$ have the same curve $\text{Im}(f) = \text{Im}(g)$

## 11.2 Class

Let $f : I \to \mathbf{R^n}$ be a path, say $f$ is of class $C^{(k)}, k \in \mathbf{N}$, and $f$ is differentiable $k-$times and *derivatives are continuous*

**Example:** $y^2 = x^3$

Parametrized: $f(t) = (t, t^{3/2}) \to f'(t) = \left(1, \frac{3}{2} \cdot \sqrt{t}\right) \to f'' = \left(0, \frac{3}{4} \cdot \frac{1}{\sqrt{t}}\right)$, which is not defined at $t = 0$

$\therefore f$ is of class $C^1$ and not $C^2$

# 12 Arc Length, Divergence and Curl

## 12.1 Arc Length

Arc length from $a$ to $b$, with $f : I \to \mathbb{R}^m$, and $c$ is a curve in $f$:

$L(f) = \int_a^b ||f'(t)||\, dx$

Method: get parametrization $f(t)$, get speed, then integrate w.r.t. bounds

## 12.2 Divergence of a vector field

Denoted by $Div(f) = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$, it can measure net mass flow or flux density

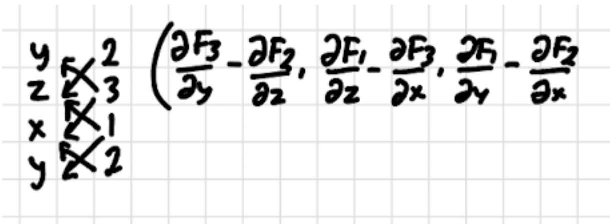
If $Div(f) > 0$, consider the field as a source, flowing out If $Div(f) < 0$, consider the field as a sink, flows in

## 12.3 Curl of a vector field

Let $F = (F_1, F_2, F_3)$ be a differentiable vector field in $\mathbb{R}^3$

$$Curl(F) = \nabla \times F = \begin{vmatrix} i & -j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$Curl(F) = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial x}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$



# 13 Gradient, Divergence and Curl (cont.)

Scalar field: $f(x, y, z)$ Vector field: $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\nabla f$ inputs a scalar field and outputs a vector field

$\nabla \cdot F$ inputs a vector field $\vec{F}$ and outputs a scalar field

$\nabla \times F$ inputs a vector field $\vec{F}$ and outputs a vector field

## 13.1 Identities

The curl of a gradient, $\nabla \times (\nabla f) = \vec{0}$, gradient fields are irrotational

The divergence of a curl, $\nabla \cdot (\nabla \times \vec{F}) = 0$, curl fields have no net source

The divergence of a gradient, $\nabla \cdot (\nabla f)$ is the Laplacian, $\Delta f$, a scalar field

The curl of a divergence, $\nabla \times (\nabla \cdot \vec{F})$ is undefined, divergence can't input a scalar field

The gradient of a curl, $\nabla(\nabla \times \vec{F})$ is undefined, gradient can't input a vector field

The curl of a curl, $\nabla \times (\nabla \times (F)) = \nabla(Div(F)) - \nabla^2 F$, and is defined in $\mathbb{R}^3$

$G$ is conservative if $\exists f : U \to \mathbb{R}$ such that $G = \nabla F$, where $F$ is the potential function

The dot product of two vector fields, e.g. $F \cdot G$, is a scalar field defined by $\mathbb{R}^3 \to \mathbb{R}$

If $G : U(\subseteq \mathbb{R}^2) \to \mathbb{R}^2$, so $(G_1, G_2)$. If $Curl(G) = 0$, then $G$ is conservative

If $G : U(\subseteq \mathbb{R}^3) \to \mathbb{R}^3$, if $G$ is the curl of some vector field, then div=0

# 14 Special Domains and Conservative Functions

Let $U \subseteq \mathbb{R}^3$ be an open set

$U$ is simply connected if:

1. $U$ is connected (any two points can be connected by a path)
2. Every loop inside $U$ can be shrunk continuously to a point inside $U$

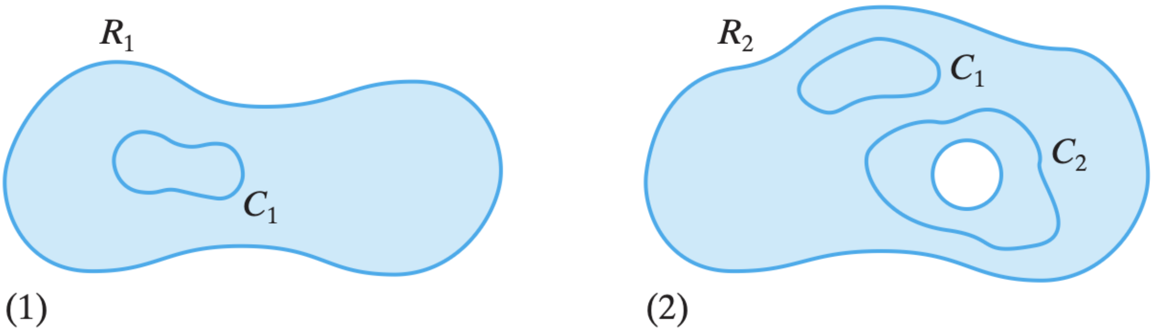



**Figure 6.35** (1) The region $R_1 \subset \mathbf{R}^2$ is simply-connected: All points surrounded by any simple, closed curve in $R_1$ lie in $R_1$. (2) In contrast, $R_2$ is not simply-connected: Although the curve $C_1$ encloses points that lie in $R_2$, the curve $C_2$ surrounds a hole. Hence, $C_2$ cannot be continuously shrunk to a point while remaining in $R_2$.

If we let $U \subseteq \mathbb{R}^n$ be a simply connected open set, and $F : U \to \mathbb{R}^n$ be a vector field, then $f$ is conservative if and only if $Curl(f) = 0$

**Example:**

Let $G(x, y, z) = (y^2, 2xy + z, y - \sin z)$, is $G$ conservative? If so, find the potential function $f$ such that $G = \nabla f$

Domain$(G) = \mathbb{R}^3$, simply connected, and $Curl(G) = (1 - 1, 0, 0 - 2y - 2y) = 0$, thus $G$ is conservative

Let $(G_1, G_2, G_3) = (F_x, F_y, F_z)$

$F_x = y^2 \Rightarrow \int F_x dx = xy^2 + g(y, z)$

$F_y = 2xy + z \Rightarrow \frac{\partial F(x,y,z)}{\partial y} = 2xy + \frac{\partial g(y,z)}{\partial y} = 2xy + z \Rightarrow \frac{\partial g(y,z)}{\partial y} = z$

$g(y, z) = \int zdy = yz + h(z) \Rightarrow F(x, y, z) = xy^2 + yz + h(z)$

$F_z = y - \sin z \Rightarrow \frac{\partial F(x,y,z)}{\partial z} = y + \frac{dh(z)}{dz} = y - \sin z \Rightarrow \frac{dh(z)}{dz} = -\sin z$

$h(z) = \int - \sin zdz = \cos z + C$

$\therefore F(x, y, z) = xy^2 + yz + \cos z$

# 15 Riemann Sums

## 15.1 Single-variable Integration

Let $f[a,b] \to \mathbb{R}$ be a function

$\int_a^b f(x)\,dx$ represents the area under the curve

We partition $[a,b]$ into subintervals for **Riemann sums**

Area under $f \approx$ sum of area of rectangles, $A = f(\xi_i)\delta x_i, \Delta x_i = (a_i - a_{i-1}), \xi \in [a_{i-1}, a_i]$

$A$ is integrable on $[a,b]$ if $\lim\limits_{\Delta x_i \to 0} \int \Sigma_{i=1}^{n} f(\xi_i)\Delta x_i$ exists

## 15.2 How to integrate functions of two variables

Let $f : [a,b] \times [c,d] \to \mathbb{R}$

$\Delta x_i = a_i - a_{i-1}, \Delta y_j = c_j - c_{j-1}$

$V$ of partitions $= lbh = f(_\xi)\Delta x_i \Delta y_j$

$Vol(A) \approx \Sigma_{i=1}^{n}\Sigma_{j=1}^{m} f(\xi_i)\Delta x_i \Delta y_j$

$f$ is integrable over $[a,b]\times[c,d]$ if $\lim\limits_{\Delta x_i \text{ and } \Delta y_j \to 0} \Sigma_{i=1}^{n}\Sigma_{j=1}^{m} f(\xi_i)\Delta x_i \Delta y_j$ exists, and is denoted by $\iint_{[a,b]\times[c,d]} f dA$

If $f$ is continuous over $[a,b] \times [c,d]$, then it is integrable

**Fubini's Theorem:** let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous

Then, $\iint_{[a,b]\times[c,d]} = \int_c^d \int_a^b f(x,y)\,dx\,dy$ and can be reversed

# 16 Cheat Sheet

## 16.1 Delta-Epsilon

The condition $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ means our input point is inside the $\delta-$neighbourhood of $(a, b)$

The proof then shows that whenever the input point is that close to $(a, b)$, the function value $f(x, y)$ lies in the $\varepsilon-$neighbourhood of the limit $L$: $|f(x, y) - L| < \varepsilon$

**Proof:** Given $\varepsilon > 0$ We want to find $\delta > 0$ such that if $0 < ||x-a|| < \delta$, then $|f(x)-L| < \varepsilon$

Start with $|f(x) - L|$ and manipulate it to relate it to $||x - a||$ For instance, show: $|f(x) - L| \leq c||x - a||$ for some $c > 0$

Choose $\delta = \frac{\varepsilon}{c}$ and show that $|f(x) - L| < c||x - a|| < c\delta = \varepsilon$

Therefore, $\lim\limits_{x \to a} f(x) = L$

## 16.2 Disproving a Multivariable Limit

1. Prove with direct substitution

If you get a determinate value (like 5, 0, or $\infty$) and the function is built from continuous functions, you're done

If you get an indeterminate form like 0/0, proceed with next steps.

2. Disprove with two-path test

For a limit approaching $(0, 0)$, common paths to test include: axis paths (along x, let $y = 0$, vice-versa), linear paths $y = mx$ and the limit d/n exist if it depends on $m$, parabolic paths

3. Disprove with polar coordinates $x = r \cos \theta, y = r \sin \theta$

## 16.3 Partial Derivative

Definition of partial derivatives at a point

$\frac{\partial F}{\partial x}(0, 0) = \lim\limits_{h \to 0} \frac{F(h,0)-F(0,0)}{h}$

## 16.4 Derivative Matrix

$D(G \circ F)(\vec{a}) = DG(F(\vec{a}))DF(\vec{a})$

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \cdots \\ \nabla f_m \end{matrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n} \\ \cdots \\ \frac{\partial f_m}{\partial x_1}, \frac{\partial f_m}{\partial x_2}, \ldots, \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Let $A = [a, b]$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $AB = \begin{bmatrix} ae + bg & af + bh \end{bmatrix}$

Let $A$ be of size $m \times m$ and $B$ of size $p \times q$, then $C = A \times B$ has dimensions $m \times q$

## 16.5 Divergence and Curl

The divergence of $F$ denoted by $\nabla \cdot F$ is $\mathbb{R}^3 \to \mathbb{R}$, measures the net rate of flow outward from a point, and is $\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

The curl of $F$ denoted by $\nabla \times F$ is $\mathbb{R}^3 \to \mathbb{R}^3$, measures the tendency to rotate or swirl around a point, and is $\nabla \times F =< \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} >$

The gradient of $f$ denoted by $\nabla f$ is $\mathbb{R}^3 \to \mathbb{R}^3$ points in the direction of greatest increase of $f$, and its magnitude is the rate of increase, and is $\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$

$\nabla \cdot (\nabla \times F) = 0$, or in words, the divergence of the curl of any vector field $F$ is 0